1.

Answer: $\boxed{\text{see below}}$ We first note that main diagonal (the squares with row number equal to column number) is a permutation of $1, 2, \ldots, n$. This is because each number i ($1 \le i \le n$) appears an even number of times off the main diagonal, so must appear an odd number of times on the main diagonal. Thus, we may assume that the main diagonal's values are $1, 2, \ldots, n$ in that order. Call any matrix satifsying this condition and the problem conditions *good*. Let $g(n)$ denote the number of good matrices. It now remains to show that $g(n) \ge \frac{(n-1)!}{\varphi(n)}$.

Now, consider a round-robin tournament with n teams, labeled from 1 through n, with the matches spread over n days such that on day i, all teams except team i play exactly one match (so there are $\frac{n-1}{2}$ pairings), and at the end of n days, each pair of teams has played exactly once. We consider two such tournaments distinct if there is some pairing of teams i, j which occurs on different days in the tournaments. We claim that the tournaments are in bijection with the good matrices.

Proof of Claim: Given any good matrix A, we construct a tournament by making day k have matches between team i and j for each i, j such that $A_{i,j} = k$, besides $(i, j) = (k, k)$. Every pair will play some day, and since each column and row contains exactly one value of each number, no team will play more than once a day. Furthermore, given two distinct good matrices, there exists a value (off the main diagonal) on which they differ; this value corresponds to the same pair playing on different dates, so the corresponding tournaments must be distinct. For the other direction, take any tournament. Make a matrix A with the main diagonal as $1, 2, \ldots, n$, and for each k, set $A_{i,j} = k$ for each i, j such that teams i, j play each other on day k. This gives a good matrix. Similarly, given any two distinct tournaments, there exists a team pair i, j which play each other on different days; this corresponds to a differing value on the corresponding good matrices.

It now suffices to exhibit $\frac{(n-1)!}{\varphi(n)}$ distinct tournaments. (It may be helpful here to think of the days in the tournament as an unordered collection of sets of pairings, with the order implictly imposed by the team not present in the set of pairings.) For our construction, consider a regular n-gon with center O. Label the points as $A_1, A_2, \ldots, A_n$ as an arbitrary permutation (so there are $n!$ possible labelings). The team k will be represented by A_k. For each k, consider the line A_kO. The remaining $n-1$ vertices can be paired into $\frac{n-1}{2}$ groups which are perpendicular to this line; use these pairings for day k. Of course, this doesn't generate $n!$ distinct tournaments– but how many does it make?

Consider any permutation of labels. Starting from an arbitrary point, let the points of the polygon be $A_{\pi(1)}, A_{\pi(2)}, \ldots, A_{\pi(n)}$ in clockwise order. Letting $\pi(0) = \pi(n)$ and $\pi(n+1) = \pi(1)$, we note that $\pi(i-1)$ and $\pi(i+1)$ play each other on day $\pi(i)$. We then see that any other permutation of labels representing the same tournament must have $A_{\pi(i-1)}A_{\pi(i)} = A_{\pi(i)}A_{\pi(i+1)}$ for all i. Thus, if $A_{\pi(1)}$ is k vertices clockwise of $A_{\pi(0)}$, then $A_{\pi(2)}$ is k vertices clockwise of $A_{\pi(1)}$, and so on all the way up to $A_{\pi(n-1)}$ being k vertices clockwise of $A_{\pi(n)}$. This is only possible if k is relatively prime to n,so there are $\varphi(n)$ choices of k. There are n choices of the place to put $A_{\pi(1)}$, giving $n\varphi(n)$ choices of permutations meeting this condition. It is clear that each permutation meeting this condition provides the same tournament, so the $n!$ permutations can be partitioned into equivalence classes of size $n\varphi(n)$ each. Thus, there are $\frac{n!}{n\varphi(n)}$ distinct equivalence classes, and we are done.

2.

$\Rightarrow$, i.e. n has ≥ 3 prime divisors: Let $n = \prod p_i^{e_i}$. Note it suffices to only consider regular p_i-gons. Label the vertices of the n-gon $0, 1, \ldots, n-1$. Let $S = \{\frac{xn}{p_1} : 0 \leq x \leq p_1 - 1\}$, and let $S_j = S + \frac{jn}{p_3}$ for $0 \leq j \leq p_3 - 2$. ($S + a = \{s + a : s \in S\}$.) Then let $S_{p_3-1} = \{\frac{xn}{p_2} : 0 \leq x \leq p_2 - 1\} + \frac{(p_3-1)n}{p_3}$. Finally, let $S' = \{\frac{xn}{p_3} : 0 \leq x \leq p_3 - 1\}$. Then I claim

$$\left(\bigsqcup_{i=0}^{p_3-1} S_i\right) \setminus S'$$

is well-centered but not decomposable. Well-centered follows from the construction: I only added and subtracted off regular polygons. To show that its decomposable, consider $\frac{n}{p_1}$. Clearly this is in the set, but isn't in S'. I claim that $\frac{n}{p_1}$ isn't in any more regular p_i-gons. For $i \geq 4$, this means that $\frac{n}{p_1} + \frac{n}{p_i}$ is in some set. But this is a contradiction, as we can easily check that all points we added in are multiples of $p_i^{e_i}$, while $\frac{n}{p_i}$ isn't.

For $i = 1$, note that 0 was removed by S'. For $i = 2$, note that the only multiples of $p_3^{e_3}$ that are in some S_j are $0, \frac{n}{p_1}, \ldots, \frac{(p_1-1)n}{p_1}$. In particular, $\frac{n}{p_1} + \frac{n}{p_2}$ isn't in any S_j. So it suffices to consider the case $i = 3$, but it is easy to show that $\frac{n}{p_1} + \frac{(p_3-1)n}{p_3}$ isn't in any S_i. So we're done.

$\Leftarrow$, i.e. n has ≤ 2 prime divisors: This part seems to require knowledge of cyclotomic polynomials. These will easily give a solution in the case $n = p^a$. Now, instead turn to the case $n = p^a q^b$. The next lemma is the key ingredient to the solution.

Lemma: Every well-centered subpolygon can be gotten by adding in and subtracting off regular polygons.

Note that this is weaker than the problem claim, as the problem claims that adding in polygons is enough.

Proof. It is easy to verify that $\phi_n(x) = \frac{(x^n-1)(x^{\frac{n}{pq}}-1)}{(x^{\frac{n}{p}}-1)(x^{\frac{n}{q}}-1)}$. Therefore, it suffices to check that there exist integer polynomials $c(x), d(x)$ such that

$$\frac{x^n - 1}{x^{\frac{n}{p}} - 1} \cdot c(x) + \frac{x^n - 1}{x^{\frac{n}{q}} - 1} \cdot d(x) = \frac{(x^n - 1)(x^{\frac{n}{pq}} - 1)}{(x^{\frac{n}{p}} - 1)(x^{\frac{n}{q}} - 1)}.$$

Rearranging means that we want

$$(x^{\frac{n}{q}} - 1) \cdot c(x) + (x^{\frac{n}{p}} - 1) \cdot d(x) = x^{\frac{n}{pq}} - 1.$$

But now, since $\gcd(n/p, n/q) = n/pq$, there exist positive integers s, t such that $\frac{sn}{q} - \frac{tn}{p} = \frac{n}{pq}$. Now choose $c(x) = \frac{x^{\frac{sn}{q}}-1}{x^{\frac{n}{q}}-1}, d(x) = \frac{x^{\frac{sn}{q}}-x^{\frac{n}{pq}}}{x^{\frac{n}{p}}-1}$ to finish. $\square$

Now we can finish combinatorially. Say we need subtraction, and at some point we subtract off a p-gon. All the points in the p-gon must have been added at some point. If any of them was added from a p-gon, we could just cancel both p-gons. If they all came from a q-gon, then the sum of those p q-gons would be a pq-gon, which could have been instead written as the sum of q p-gons. So we don't need subtraction either way. This completes the proof.

3.

Solution Ⅰ When $n \geqslant 4$, we consider the set

$$M = \{m, m+1, m+2, \cdots, m+n-1\}.$$

If $2 \mid m$, then $m+1$, $m+2$, $m+3$ are mutually prime;

If $2 \nmid m$, then m, $m+1$, $m+2$ are mutually prime.

Therefore, in every n-element subset of M, there are at least 3 mutually prime elements. Hence there exists $f(n)$ and

$$f(n) \leqslant n.$$

Let $T_n = \{t \mid t \leqslant n+1 \text{ and } 2 \mid t \text{ or } 3 \mid t\}$, then T_n is a subset of $\{2, 3, \cdots, n+1\}$. But any 3 elements in T_n are not mutually prime, thus $f(n) \geqslant |T_n| + 1$.

By the inclusion and exclusion principle, we have

$$|T_n| = \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] - \left[\frac{n+1}{6}\right].$$

Thus

$$f(n) \geqslant \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] - \left[\frac{n+1}{6}\right] + 1. \qquad (1)$$

Therefore

$$f(4) \geqslant 4, \; f(5) \geqslant 5, \; f(6) \geqslant 5,$$

$$f(7) \geqslant 6, \; f(8) \geqslant 7, \; f(9) \geqslant 8.$$

Now we prove that $f(6) = 5$.

Let x_1, x_2, x_3, x_4, x_5 be 5 numbers in $\{m, m+1, \cdots, m+5\}$. If among these 5 numbers there are 3 odds, then they are mutually prime. If there are 2 odds among these 5 numbers, then the other three numbers are even, say x_1, x_2, x_3, and the 2 odds are x_4, x_5.

When $1 \leqslant i < j \leqslant 3$, $|x_i - x_j| \in \{2, 4\}$. Thus among x_1, x_2, x_3 there is at most one which is divisible by 3, and at most one which is divisible by 5. Therefore, there is at least one which is neither divisible by 3 nor by 5, say, $3 \nmid x_3$ and $5 \nmid x_3$. Then x_3, x_4, x_5 are mutually prime. This is to say, among these 5 numbers there are 3 elements which are mutually prime, i.e. $f(6) = 5$.

On the other hand, $\{m, m+1, \cdots, m+n\} = \{m, m+1, \cdots, m+n-1\} \cup \{m+n\}$ implies that

$$f(n+1) \leqslant f(n) + 1.$$

Since $f(6) = 5$, we have

$$f(4) = 4,\ f(5) = 5,\ f(7) = 6,\ f(8) = 7,\ f(9) = 8.$$

Thus when $4 \leqslant n \leqslant 9$,

$$f(n) = \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] - \left[\frac{n+1}{6}\right] + 1. \tag{2}$$

In the following we will prove that (2) holds for all n by mathematical induction.

Suppose that equation (2) holds for all $n \leqslant k$ $(k \geqslant 9)$. In the case when $n = k+1$, since

$$\{m, m+1, \cdots, m+k\} = \{m, m+1, \cdots, m+k-6\} \cup$$
$$\{m+k-5, m+k-4, m+k-3, m+k-2, m+k-1, m+k\},$$

equation (2) holds for $n = 6$, $n = k-5$, we have

$$f(k+1) \leqslant f(k-5) + f(6) - 1$$

$$= \left[\frac{k+2}{2}\right] + \left[\frac{k+2}{3}\right] - \left[\frac{k+2}{6}\right] + 1. \tag{3}$$

By (1) and (3) we obtain that equation (2) holds for $n = k+1$. Consequently, for any $n \geqslant 4$, we have

$$f(n) = \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] - \left[\frac{n+1}{6}\right] + 1.$$

4.

Answer: $\boxed{\text{see below}}$ We write all fractions of the form b/a, where a and b are relatively prime, and $0 \le b \le a \le n$, in ascending order. For instance, for $n = 5$, this is the sequence

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

This sequence is known as the *Farey sequence.*

Now, if we look at the the sequence of the denominators of the fractions, we see that k appears $\varphi(k)$ times when $k > 1$, although 1 appears twice. Thus, there are $N + 1$ elements in the Farey sequence. Let the Farey sequence be

$$\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_{N+1}}{a_{N+1}}$$

Now, $a_{N+1} = 1$, so the sequence $a_1, a_2, \ldots, a_N$ contains $\varphi(k)$ instances of k for every $1 \le k \le n$. We claim that this sequence also satisfies

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_Na_1} = 1.$$

Since $a_1 = a_{N+1} = 1$, we have

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_Na_1} = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_Na_{N+1}}.$$

Now, it will suffice to show that $\frac{1}{a_ia_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$. Once we have shown this, the above sum will telescope to $\frac{b_{N+1}}{a_{N+1}} - \frac{b_1}{a_1} = 1 - 0 = 1$.

To see why $\frac{1}{a_ia_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$ holds, we note that this is equivalent to $1 = b_{i+1}a_i - b_ia_{i+1}$. We can prove this fact geometrically: consider the triangle in the plane with vertices $(0, 0)$, (a_i, b_i), and (a_{i+1}, b_{i+1}). This triangle contains these three boundary points, but it contains no other boundary or interior points since a_i and a_{i+1} are relatively prime to b_i and b_{i+1}, respectively, and since no other fraction with denominator at most n lies between $\frac{b_i}{a_i}$ and $\frac{b_{i+1}}{a_{i+1}}$. Thus, by Pick's theorem, this triangle has area $1/2$. But the area of the triangle can also be computed as the cross product $\frac{1}{2}(b_{i+1}a_i - b_ia_{i+1})$; hence $b_{i+1}a_i - b_ia_{i+1} = 1$ and we are done.

5.

Denote the sum from the statement by S_n. We will prove a stronger inequality, namely,

$$S_n > \frac{n}{2}(\log_2 n - 4).$$

The solution is based on the following obvious fact: no odd number but 1 divides 2^n evenly. Hence the residue of 2^n modulo such an odd number is nonzero. From here we deduce that the residue of 2^n modulo a number of the form $2^m(2k+1)$, $k > 1$, is at least 2^m. Indeed, if $2^{n-m} = (2k+1)q + r$, with $1 \le r < 2k+1$, then $2^n = 2^m(2k+1)q + 2^m r$, with $2^m < 2^m r < 2^m(2k+1)$. And so $2^m r$ is the remainder obtained by dividing 2^n by $2^m(2k+1)$.

Therefore, $S_n \ge 1\times$(the number of integers of the form $2k+1, k > 1$, not exceeding n)$+2\times$(the number of integers of the form $2(2k+1)$, $k > 1$, not exceeding n)$+2^2\times$(the number of integers of the form $2^2(2k+1)$, $k > 1$, not exceeding n)$+\cdots$.

Let us look at the $(j+1)$st term in this estimate. This term is equal to 2^j multiplied by the number of odd numbers between 3 and $\frac{n}{2^j}$, and the latter is at least $\frac{1}{2}(\frac{n}{2^j} - 3)$. We deduce that

$$S_n \ge \sum_j 2^j \frac{n - 3\cdot 2^j}{2^{j+1}} = \sum_j \frac{1}{2}\left(n - 3\cdot 2^j\right),$$

where the sums stop when $2^j \cdot 3 > n$, that is, when $j = \lfloor \log_2 \frac{n}{3} \rfloor$. Setting $l = \lfloor \log_2 \frac{n}{3} \rfloor$, we have

$$S_n \ge (l+1)\frac{n}{2} - \frac{3}{2}\sum_{j=0}^{l} 2^i > (l+1)\frac{n}{2} - \frac{3\cdot 2^{l+1}}{2}.$$

Recalling the definition of l, we conclude that

$$S_n > \frac{n}{2}\log_2\frac{n}{3} - n = \frac{n}{2}\left(\log_2\frac{n}{3} - 2\right) > \frac{n}{2}(\log_2 n - 4),$$

and the claim is proved. The inequality from the statement follows from the fact that for $n > 1000$, $\frac{1}{2}(\log_2 n - 4) > \frac{1}{2}(\log_2 1000 - 4) > 2$.

6.

The number of subsets with the sum of the elements equal to n is the coefficient of x^n in the product

$$G(x) = (1+x)(1+x^2)\cdots(1+x^p).$$

We are asked to compute the sum of the coefficients of x^n for n divisible by p. Call this number $s(p)$. There is no nice way of expanding the generating function; instead we compute $s(p)$ using particular values of G. It is natural to try pth roots of unity.

The first observation is that if ξ is a pth root of unity, then $\sum_{k=1}^{p} \xi^p$ is zero except when $\xi = 1$. Thus if we sum the values of G at the pth roots of unity, only those terms with exponent divisible by p will survive. To be precise, if ξ is a pth root of unity different from 1, then

$$\sum_{k=1}^{p} G(\xi^k) = ps(p).$$

We are left with the problem of computing $G(\xi^k)$, $k = 1, 2, \ldots, p$. For $k = p$, this is just 2^p. For $k = 1, 2, \ldots, p-1$,

$$\begin{aligned} G(\xi^k) &= \prod_{j=1}^{p}(1+\xi^{kj}) = \prod_{j=1}^{p}(1+\xi^j) = (-1)^p \prod_{j=1}^{p}((-1)-\xi^j) = (-1)^p((-1)^p - 1) \\ &= 2. \end{aligned}$$

We therefore have $ps(p) = 2^p + 2(p-1) = 2^p + 2p - 2$. The answer to the problem is $s(p) = \frac{2^p-2}{p} + 2$. The expression is an integer because of Fermat's little theorem.

7.

Answer: Maximum: **9**, minimum: **-10**, number of terms: **346**.

Calculating the first few values, we find: (n is in binary)

n	a_n
1	0
10	-1
11	1
100	0
101	-2
110	0
111	2
1000	1
1001	-1
1010	-3
1011	-1
1100	1
1101	-1
1110	1
1111	3

If the last two digits are 00 or 11, then a_n is one more than $a_{n/2}$ If the last two digits are 01 or 10, then a_n is one less than $a_{n/2}$. Note that $[\frac{n}{2}]$ is formed by deleting the last binary digit of n. So a_n =(number of adjacent pairs 00 and 11) - (number of adjacent pairs 01 and 10). For example, 1100 has pairs 11, 10 and 00, so a_{12} = 2 - 1 = 1.

1996 = 11111001100, so the maximum value of a_n for n 1996 is at n = 1111111111 = 1023 with value 9. Similarly, the minimum value is at n = 10101010101 with value -10.

If $a_n = 0$, then n must have an odd number of binary digits, with the first digit 1. The only 1 digit number is $n = 1$. The 3 digit numbers (with $a_n = 0$) are 110 and 100. The 5 digit numbers are 11101, 11001, 11011, 10111, 10001 and 10011. Consider the $2m+1$ digit numbers. Exactly m of the digits after the initial 1 must be such that they are the same as the previous digit. Specifying those digits completely determines the number, so there are $\binom{2m}{m}$ such numbers (where $\binom{a}{b}$ is the binomial coefficient). Thus there are $\binom{6}{3}$ 7-digit numbers, $\binom{8}{4}$ 9-digit numbers and $\binom{10}{5}$ 11-digit numbers. But the 11-digit numbers 11111100000, 11111010100, 11111010110, 11111010010, 11111011010 are greater than 1996. Hence the required number is 1 + 2 + 6 + 20 + 70 + 252 - 5 = 346.